same magnitude but opposite sign. It then makes little difference whether $[M_o]$ is lumped and $[M_R]$ is consistent or vice versa. A usually conservative interpretation is that the average value $(\omega_{oi}+\omega_{Ri})/2$ is in error by e_i, at most.

Numerical results for method 2 appear in Table 2. The beam problem is omitted because its moment field is already interelement-continuous and would therefore yield error estimates of zero. Accuracy is satisfactory if ω_{Ri} is reasonably accurate, as usually happens if $[M_R]$ is the consistent mass matrix.

The foregoing results are encouraging, but substantiation from larger and more complicated problems is desirable.

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Generalized Kelvin Function Solutions for a Class of Vibrating Circular-Plate Problems

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Introduction

THE study of the vibration of a circular plate finds many important applications. In this Note, we consider a class of vibration problems of a circular plate subjected to a periodic excitation force and a viscous damping force. Two examples are the forced vibrations of 1) the injector plate due to pressure oscillation in the combustion chamber of a liquid rocket engine such as the Space Shuttle main engine,[1] and 2) the circular lid of an underwater container subjected to water motion. The governing equation for the transverse deflection of the plate is[2]

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} = p \tag{1}$$

where

$$D = Eh^3/[12/(1-\nu^2)] \tag{2}$$

is the flexural rigidity; E is the Young's modulus; h is the plate thickness; ν is the Poisson ratio; $\nabla^4 = \nabla^2\nabla^2$ is the biharmonic differential operator; and for the circular plate, it is expressed in polar notation:

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial^2\theta}\right)^2 \tag{3}$$

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In Eq. (1), w is the transverse deflection of a typical point on the plate; ρ is the mass per unit volume; t is the time; c is the viscous damping coefficient; and p is the transverse excitation force.

With zero values of c and p and homogenous boundary conditions, Eq. (1) constitutes an eigenvalue problem for the vibration of the plate. The classical method for structural vibrations is by eigenfunction superposition,[2] whereas the present paper solves the structural vibration as a boundary value problem in a straightforward manner. It yields a closed-form (instead of infinite series) exact solution, which is expressed in terms of the first and second kinds of Bessel functions with complex arguments. No formulas are available for these types of functions. This Note introduces a new set of modified Bessel functions for the complex variables, called functions H and Q. By using this new set of functions, the displacement, velocity, internal force, or bending moments of the elastic vibration of the circular plate can be readily calculated.

Formulation and Solution

Since we are considering a periodic excitation force, it is possible to expand both the force p and displacement function w in a complex form of the Fourier series:

$$p = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{mn}(r)\, e^{i(m\theta + \omega n t)}$$

$$w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} w_{mn}(r) e^{i(m\theta + \omega n t)} \tag{4}$$

where m is an integer representing the mth transverse mode of plate vibration, θ is the angular coordinate measured in the circumferential direction of the plate, and ω is the first harmonic frequency of the periodic force.

Now Eq. (1) becomes

$$D\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)^2 w_{mn} - \rho h(\omega n)^2 w_{mn} + i\, c\omega n\, w_{mn} = p_{mn} \tag{5}$$

First, let us find the complementary solution of Eq. (5) in the form

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)^2 w_{nnc} - k_n^4 w_{mnc} = 0 \tag{6}$$

where

$$k_n = \left\{\rho h(\omega n)^2 \frac{[1 - i\,c/(\rho h\omega n)]}{D}\right\}^{1/4} = s_n \exp\left(\frac{i\psi}{4}\right) \tag{7a}$$

$$s_n = \left\{\rho h(\omega n)^2 \frac{[1 + (\mu/n)^2]^{1/2}}{D}\right\}^{1/4} \tag{7b}$$

$$\mu = c/(\rho h\omega), \qquad \psi = \arctan(-\mu/n) \tag{7c}$$

Equation (6) can be further separated into

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right) w_{mnc} + k_n^2 w_{mnc} = 0 \tag{8a}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right) w_{mnc} - k_n^2 w_{mnc} = 0 \tag{8b}$$

The solution to Eq. (8a) is in terms of the mth-order Bessel functions of the first and second kinds, $J_m(k_n r)$ and $Y_m(k_n r)$, and that of Eq. (8b) is in terms of the mth-order modified Bessel functions of the first and second kinds, $I_m(k_n r)$ and $K_m(k_n r)$. The solution to Eq. (5) is therefore

$$w_{mn} = C_1\, J_m(k_n r) + C_2\, Y_m(k_n r) + C_3\, I_m(k_n r) + C_4\, K_m(k_n r) + w_{mnp} \tag{9}$$

where w_{mnp} represents a particular solution to Eq. (5). Since the value of w is finite at $r = 0$, the integration constants C_2 and C_4 in Eq. (9) are set to zero. The constants C_1 and C_3 are determined from the boundary conditions at $r = R$. For example, for a clamped edge,

$$(w)_{r=R} = 0, \qquad \left(\frac{\partial w}{\partial r}\right)_{r=R} = 0 \tag{10a}$$

and for a simple supported edge,

$$(w)_{r=R} = 0, \qquad \left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial^2 \theta}\right)\right]_{r=R} = 0 \tag{10b}$$

One of the obstacles to finding the deflection w in expression (9) is the complex argument of the Bessel functions due to k_n. There are no formulas or tables available to determine them. Here we introduce a new set of functions, called H and Q, which are used to separate the Bessel functions of complex argument into real and imaginary parts. For a complex variable z, the functions H and Q satisfy

$$J_m(z) = Hr_m(z) + i\,Hi_m(z) \tag{11a}$$

$$I_m(z) = Qr_m(z) + i\,Qi_m(z) \tag{11b}$$

One may realize that when $z = i^{3/2}x$, the present Hr and Hi recover the well-known Kelvin functions of *Ber* and *Bei*, respectively. In this sense, the H and Q functions just defined are the generalized Kelvin functions. By expanding on Bessel functions I_m and J_m and letting

$$z = \zeta e^{i\gamma} \tag{12}$$

one can find

$$Hr_m(\zeta e^{i\gamma}) = \sum_{j=0}^{\infty}(-1)^j\left(\frac{\zeta}{2}\right)^{m+2j}\frac{\cos[(m+2j)\gamma]}{j!\Gamma(m+j+1)} \tag{13a}$$

$$Hi_m(\zeta e^{i\gamma}) = \sum_{j=0}^{\infty}(-1)^j\left(\frac{\zeta}{2}\right)^{m+2j}\frac{\sin[(m+2j)\gamma]}{j!\Gamma(m+j+1)} \tag{13b}$$

$$Qr_m(\zeta e^{i\gamma}) = \sum_{j=0}^{\infty}\left(\frac{\zeta}{2}\right)^{m+2j}\frac{\cos[(m+2j)\gamma]}{j!\Gamma(m+j+1)} \tag{13c}$$

$$Qi_m(\zeta e^{i\gamma}) = \sum_{j=0}^{\infty}\left(\frac{\zeta}{2}\right)^{m+2j}\frac{\sin[(m+2j)\gamma]}{j!\Gamma(m+j+1)} \tag{13d}$$

Once the Bessel functions in expression (9) are separated into H and Q functions, the calculation for deformation w becomes straightforward.

Sample Results

Uniformly Distributed Force

First, we consider the case in which p is uniformly distributed and sinusoidally varies with time. The typical application of this problem is the forced vibration of an injector plate due to combustion instability, as in the Space Shuttle main engine.[1] Here, the solution to Eq. (5) is axisymmetric, and m and n are 0 and 1, respectively. This particular solution can be easily found:

$$w_{01p} = -p_{01}/[\rho h(\omega)^2 - i\,c\,\omega] \tag{14}$$

with p_{01} as the amplitude of the sinusoidal force. If the plate is subjected to a clamped boundary condition at $r = R$, by substituting Eq. (9) into Eq. (10a), the constants C_1 and C_3 are determined, and so is the solution:

$$w = p_{01}\frac{[I_1(a_1)J_0(k_1r) + J_1(a_1)I_0(k_1r) - \Delta]}{[\Delta(\rho h(\omega)^2 - i\,c\,\omega)]} \tag{15}$$

where

$$\Delta = I_1(a_1)J_0(a_1) + J_1(a_1)I_0(a_1) \tag{16a}$$

$$a_1 = k_1R = \beta(1 - i\mu), \qquad \beta = \frac{\omega^{1/2}}{[D/(\rho h R^4)]^{1/4}} \tag{16b}$$

Here, β is a dimensionless frequency and μ, as defined in Eq. (7c), is a damping coefficient. For a harmonically applied force, an interesting parameter is the dynamic magnification factor (DMF), which is the ratio of the resultant response amplitude to the static displacement that would have been produced by the force p_{01}. For the present boundary condition and applied force, the static deflection of the plate has been provided in a classic text book,[2] and the DMF is expressed as

$$\text{DMF} = 64\,\frac{[I_1(a_1)J_0(a_1r^*) + J_1(a_1)I_0(a_1r^*) - \Delta]}{[\Delta a_1^4(1 - r^{*2})^2]}\,e^{i\omega t} \tag{17}$$

where $r^* = r/R$ is a dimensionless radius. Equation (17) indicates that the DMF depends on β, μ, r^*, and ωt. The most important factor is the maximum amplitude of DMF in each cycle. This can be calculated by applying H and Q functions to the Bessel functions in Eq. (17). Shown in Fig. 1 is the variation of the maximum amplitude of the DMF with frequency β at the center of the plate ($r^* = 0$) and at several damping coefficients of μ. The responses at other radial locations are similar to the one at the center and are therefore not shown here.

It is generally observed from Fig. 1 that at low frequencies (low β), the DMF is equal to 1, which represents a spontaneous response of the plate to force. With an increase in the forcing frequency (β), the DMF exhibits several local peaks, which correspond to the resonance between the applied force and the natural vibration of the plate itself. For a system with a low damping coefficient, the resonance can be destructive and can lead to the violent vibration and even failure of the structure. The peaks in the DMF (or the damage of the resonance) can be "smeared" by viscous damping, as seen from the lines with a high value of μ. In general, the level of damage to the plate is minor at both very low and very high frequencies, and is most severe at a frequency close to the first natural frequency of the plate.

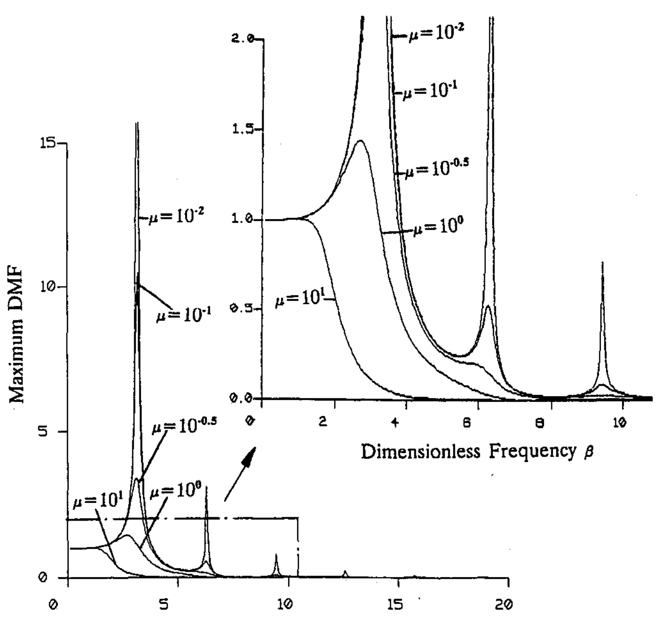
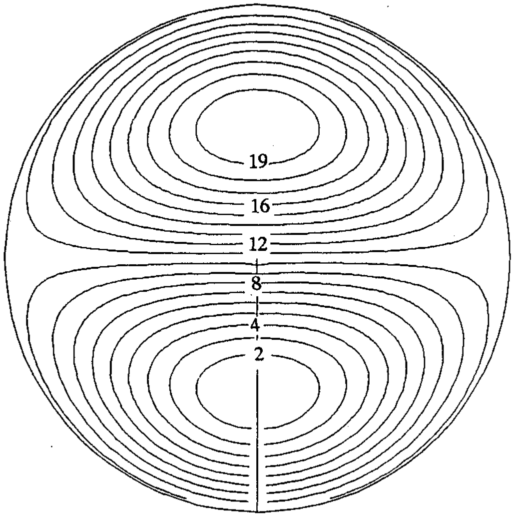


Fig 1 Dynamic magnification factor (DMF) of a circular plate subjected to a uniformly distributed, sinusoidally varying force.



CONTOUR LEVELS

1	-2.429E+00	9	-3.836E-01
2	-2.174E+00	10	-1.279E-01
3	-1.918E+00	11	1.279E-01
4	-1.662E+00	12	3.836E-01
5	-1.406E+00	13	6.393E-01
6	-1.151E+00	14	8.950E-01
7	-8.950E-01	15	1.151E+00
8	-6.393E-01	20	2.429E+00

Fig. 2 Dimensionless deflection of a circular plate subjected to a linearly distributed, sinusoidally varying force.

Linearly Distributed Force

The second example considered is the case in which the force is a linear function of y [or $r\cos(\theta)$], but still sinusoidally varies with time. This situation can be realized for an underwater circular lid on a container, when the top surface experiences a wave motion. Now m and n are all equal to 1, so that the corresponding amplitude of the oscillatory force p_{11} is nonzero. If a simple supported boundary condition [Eq. (10b)] is assumed, one can derive the dynamic deflection (w_d) as

$$w_d = p_{11}\cos(\theta)\,\frac{[c_1 I_1(a_1 r^*) + c_2 I_2(a_1 r^*) - \Delta r^*]}{[\Delta(\rho h(\omega)^2 - ic\omega)]}\,e^{i\omega t} \tag{18}$$

where

$$c_1 = a_1 J_1(a_1) + (\nu - 1)J_2(a_1), \quad c_2 = a_1 I_1(a_1) + (\nu - 1)I_2(a_1) \tag{19a}$$

$$\Delta = c_1 I_1(a_1) + c_2 J_1(a_1) \tag{19b}$$

The static deflection (w_s) is[2]

$$w_s = p_{11}\cos(\theta)R^4 r^*(1 - r^{*2})\,\frac{[7 + \nu - (3 + \nu)r^{*2}]}{[192(3 + \nu)D]} \tag{20}$$

It is found that the DMF, defined as w_d/w_s, is independent of $\cos(\theta)$, and its variation with frequency β and damping coefficient μ is similar to that shown in Fig. 1. The exception is that the natural frequencies of the plate are now slightly greater due to a different boundary condition. In Fig. 2, the dimensionless deflection of the plate $[w_d/(p_{11}R^4/D)]$ is shown at an instance when $\omega t = \pi/2$. The maximum deflection, as seen, is located at around $r^* = \frac{1}{2}$.

Conclusion

Two examples are used to demonstrate the use of H and Q functions for a class of plate vibration problems, in which the excitation force is periodic and viscous force is proportional to the velocity of plate deflection. The functions can also be applied to study the spatial instability[3] of a liquid jet.

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